Numerical Solution of Nonlinear Fractional Differential Equations with Variable Coefficients by Jacobian Iterative Method

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Abstract: When solving nonlinear fractional differential equations with Jacobian iterative method has the characteristics of keeping the iterative matrix unchanged compared with direct solution. The calculation program is generally simple, and the convergence speed is fast for many problems. By solving the two-point boundary value problem in the physical model, some numerical values are obtained to compare the iterative effects of the two iterative methods. This paper verifies some existing conclusions based on the Jacobian iterative method, and uses the designed program to solve the problem of equations.

1. Introduction

The development of Modern Applied Mathematics requires the mathematical ability to analyze and control objective phenomena to develop towards a high and accurate level with overall significance, so as to continuously accumulate the results of nonlinear analysis and gradually form an important branch of Modern Analytical Mathematics - nonlinear functional analysis [1]. Nonlinear functional analysis is a research discipline with both profound theory and wide application in mathematics, Based on the nonlinear problems in mathematics and natural science, some general theories and methods for dealing with nonlinear problems are established. Because it can well explain various natural phenomena in nature, it has great application in practical production and life, and has a positive impact on the development of physics, chemistry, bioscience, astronomy and other related disciplines, In recent years, it has been highly valued by the international mathematical and natural science circles, such as function interpolation and fitting [2], numerical solution of differential equations, and so on. Finally, it should be reduced to solving Nonlinear differential equation a x = b[3]. Therefore, how to use computer as a powerful calculation tool to solve linear equations is a very practical and important problem.

2. Brief introduction of Jacobian and iterative method

Different from the direct method, the iterative method does not obtain the solution of the equations through the predetermined finite step arithmetic, but uses a certain limit process to approach the exact solution of AX = B, that is, starting from an initial vector x (0) [4], generate an X (0), X (1), X (2),..., and make it converge to the exact solution of AX = B according to a certain iterative format. The general calculation formula of X (K + 1) is: X (K + 1) = FK (x (k), X (k-1),..., X (K-M), k = 0, 1,..., where x (K + 1) is related to X (k), X (K + 1), X (K + m), which is called multi-step iterative method. If x (K + 1) is only related to X (k), Ji is x (K + 1) = fkx (k), k = 0, 1,..., which is called single-step iterative method. Let FK be linear, that is, X (K + 1) = BKX (k) + FK, k = 0, 1,..., where BK ∈ R (n * n) ″It is called one-step linear iterative method, BK is called iterative matrix, if it is independent of BK and FK, that is, X (K + 1) = BKX (k) + FK, k = 0,1 is called one-step constant linear iterative method [5]. This paper mainly discusses the iterative method with this form, such as formula (1).

$$A=\begin{pmatrix} a_{11} & ... & a_{1n} \\ ... & ... & ... \\ a_{n1} & ... & a_{nn} \end{pmatrix}, b=\begin{pmatrix} b_1 \\ b_2 \\ ... \\ b_n \end{pmatrix} \tag{1}$$

Nonsingular, satisfying the equation AX = B

$$L=\begin{pmatrix} 0 & 0 & ... & 0 \\ a_{21} & 0 & ... & 0 \\ a_{31} & a_{32} & ... & 0 \\ a_{n1} & a_{n2} & a_{nn-1} & 0 \end{pmatrix}, U=-\begin{pmatrix} 0 & a_{12} & a_{13} & 0 \\ 0 & ... & a_{23} & 0 \\ 0 & 0 & ... & 0 \\ 0 & 0 & 0 & a_{n-\ln} \end{pmatrix} \tag{2}$$

That is, where d = diag (a11, A12,... ANN), and - L and - u are the strictly lower and upper triangular parts of a, respectively, and a = d-l-u.

According to the above conditions, let d be nonsingular, that is, aii ≠ 0, I = L,..., N. let m = D, n = L + U, which is equivalent to x = D (L + U) x + dB, so as to construct the iterative method X (k-1) = bjx (k) + F, k = 0, l,..., where the vector F and iterative matrix BJ are f = D-1, B, BJ = D (L + U) = i-da, which is called Jacobian iterative method, abbreviated as J method [6].

3. A new iterative method for nonlinear differential equations based on Jacobian approximation

3.1. Convergence based on its iterative principle

Numerical calculus theory is a theory to study the mathematical properties and applications of arbitrary order differential and integral [6]. Fractional calculus operator is nonlocal, which is very suitable for describing materials with memory and genetic properties in real life. In many complex systems, the diffusion process no longer presents Gaussian distribution, and the corresponding second law can not describe the relevant transmission behavior, while the secondary diffusion motion is particularly important in the transmission of abnormal fragment geometry, porous water seepage system and other problems [7]. According to the iterative principle of iterative method, it can be known that G-S converges faster than Jacobi iterative method. However, the analytical contain complex special functions, and many equations cannot obtain analytical solutions, so it is necessary to study the numerical solutions of fractional differential equations [8]. Numerical solutions of Nonlinear Differential Equations is very similar to that of classical differential equations. Consider the following general functional equation (3):

$$u(\bar{x}) = f(\bar{x}) + N(u(\bar{x})) \tag{3}$$

Here, it is a nonlinear equation in Banach space B → B and F are known functions, - x = (x1, X2,..., xn). In equation (3), the solution of the series form of function u is (4).

$$u(\bar{x}) = \sum_{i=0}^{\infty} u_i(\bar{x}) \tag{4}$$

The nonlinear differential equation n is decomposed into formula (5)

$$N(\sum_{i=0}^{\infty} u_i) = N(u_0) + \sum_{i=1}^{\infty} \left\{ N(\sum_{i=0}^{i} u_i) - N(\sum_{i=0}^{i-1} u_i) \right\} \tag{5}$$

Define mutual transfer relationship (6):

$$\begin{cases} u_0 = f \\ u_1 = N(u_0) \\ u_{m+1} = N(u_0 + ... + u_m) - N(u_0 + ... + u_{m-1}); m = 1, 2, ..., \end{cases} \quad (6)$$

3.2. Based on its iterative principle, the convergence rate

The Jacobian iterative method actually used should be the convergent Jacobian iterative method. Here we want to give the conditions for judging the convergence. Let X be the solution of x = BX + F and record the error vector as e (k) = x (k) - X. Then E (K + 1) = be (k), k = 0, 1,..., from which e (k) = be (0), where e (0) = x (0) - x is independent of K, so the convergence of the iterative method means that lime (k) = Limbe (0) = 0V ∈ RN. R(B)=InP(B). It is called the asymptotic convergence rate of iterative method, or the asymptotic convergence rate. Obviously, R (b) = limr (b), and R (b) has nothing to do with what norm B takes and the number of iterations. It reflects the asymptotic property of the iterative method when the number of iterations tends to infinity. In order to meet the requirements, LN / R (b) it can be used to change the minimum number of iterations of the decomposition of differential equations . It should be noted that the Jacobian iterative method used in this paper is only applicable to matrices with non-zero elements on the diagonal, so it has some limitations. However, these two iterative methods are usually used in solving large sparse linear equations, especially differential equations. In addition, these two iterative methods are often used in engineering calculation, so the study of these two methods is helpful to solve practical problems. It should be noted that the Jacobian iterative method used in this paper is only applicable to matrices with non-zero elements on the diagonal, so it has some limitations.From the above discussion, we can also see that for different iterative methods, the one with smaller spectral radius of iterative matrix converges faster.The difference and relationship between fractional calculus and integer calculus are summarized. Its essence is that fractional calculus is more general than integer calculus. Because the new value obtained by G-S iterative method in each iteration can be quickly used to calculate the next value, the number of calculations can be reduced and time can be saved.

It can be seen from the above theorem that the convergence of the iterative sequence depends on the spectral radius of the iterative matrix and is independent of the selection of the initial vector. Because the spectral radii of J-method iterative matrix and g-s-method iterative matrix are not necessarily the same when solving the same equation system, and there is no inclusion relationship, sometimes J-method iteration converges and G-S iteration does not converge. Of course, there are also cases where the J-method iteration does not converge and the G-S iteration converges [9].

A computer oriented algorithm with good computational complexity and reliable theoretical analysis is a good algorithm. The so-called computational complexity includes two aspects: time complexity and space complexity. Under the same accuracy, those with less computational time are good in time complexity, while those with less memory space are good in space [10]. The component form of Jacobian iterative method is (7).

$$x_i^{(k+1)} = \frac{1}{a_{ii}}(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)}) i = 1, 2, ..., \quad (7)$$

According to the principle that the convergence speed is faster when the spectral radius is small, it can also be further confirmed that the iterative method Jacques converges faster than the iterative method Jacques.

4. Nonlinear differential equations based on Jacobian iterative method

4.1. Numerical approximation of fractional partial derivatives

Fractional calculus theory is a theory about derivatives and integrals of any order. It is a generalization of classical integral calculus. Like integer calculus theory, fractional calculus theory also plays an important role [11]. In nature, many complex phenomena need to construct nonlinear equations and introduce some conditions that are inconsistent with the reality, or new models need

to be constructed due to the small changes of materials and external conditions. These models are very difficult to obtain both analytical and numerical solutions. Here we only give a brief introduction to several commonly used definitions of fractional derivative. We know that the multiple integration of a function can be simplified to formula (8).

$$D_t^{-n} y(t) = \frac{1}{(n-1)} \int_0^t (t-\tau)^{n-1} y(\tau) d\tau \tag{8}$$

Fractional (order) differential equations refer to differential equations with arbitrary order derivatives. It extends the integer differential equation or corresponding integral equation to any order differential equation or integral equation, and gradually develops into an important branch of differential equation [12].This method deals with the nonlinear reaction term well, and obtains a higher accurate approximate solution compared with the nonlinear fractional variable coefficient differential equation. The effectiveness of the method is verified numerically.

In the past few years, many authors have studied the Existence of positive solutions of Jacobian iterative method equations in infinite field by using different methods.

4.2. Antimissile proof

If f (y, x, t) satisfies the Jacobian iterative condition with respect to y, that is, there is a constant l such that (9)

$$D_t^{-n}(y,x,t) = \frac{1}{(n-1)} \int_0^t (y,x,t)^{n-1} y(t) dt \tag{9}$$

The order of fractional derivative is usually real valued constant or complex valued constant, but it can also be a function of time or space variables. In some models, variable fractional derivatives change with time and space. At the same time, they are also widely used as models to describe physical or chemical phenomena in some fields. For this kind of nonlinear variable fractional diffusion equation, an explicit difference scheme is given, and the corresponding theoretical proof of stability and convergence is given. Then the numerical solution is obtained through a numerical example, and then the effectiveness of the algorithm is further verified by comparing the relative error between values. Based on Jacobian iterative method, the nonlinear Jacobian iterative method diffusion differential equation (10) is satisfied

$$\left| f(u_1,x,t) - f(u_2,x,t) \right| \le L \left| u_1 - u_2 \right| \tag{10}$$

Thus, the solution of the nonlinear variable order fractional diffusion equation exists and is unique.

4.3. Numerical solution of differential equations by Jacobian iterative method

A mathematical method for approximate analytical solution of nonlinear rational equations, which is characterized by wide application range, simple calculation process and fast convergence speed. It does not need any approximate conditions. It does not need to use linear, perturbation, iteration or simplified model equations to deal with strongly nonlinear problems, nor does it need numerical solution. The main contents of the decomposition method include: (1) decompose a true solution into the sum of several solution components, try to find the solution components of each order respectively, and then let the sum of these solution components approach the true value with any required high precision. (2) The whole equation is properly decomposed into several parts. It is mainly decomposed into linear, nonlinear, deterministic and stochastic parts according to the operator. In principle, it can be decomposed arbitrarily, but pay attention to skills. For example, the linear operator of the selected deterministic term is reversible, so it is easy to obtain the partial solution of the corresponding equation of the linear operator, and then use the known initial value or boundary value conditions, The most important thing is to make the high-order solution component only depend on the low-order solution component, so that any high-order solution component can be derived from the low-order solution component according to certain rules. (3) An ingenious

method is proposed for the most important nonlinear term in the nonlinear equation to produce a polynomial equivalent to it, using a special regular polynomial (11)

$$y_{app}(x_0) = y_0(x_0) + \int_0^{x_0} \lambda(Ly_0 + Ny_0)dx \tag{11}$$

Instead of the nonlinear function, the polynomial is only determined by the previous lower order solution component and the nonlinear function.

5. Conclusions

This paper mainly discusses the use of Jacobian iterative method to solve large sparse linear equations. According to the iterative principle of iterative method, it can be known that G-S converges faster than Jacobi iterative method. The difference and relationship between fractional calculus and integer calculus are summarized. Its essence is that fractional calculus is more general than integer calculus. Because the new value obtained by G-S iterative method in each iteration can be quickly used to calculate the next value, the number of calculations can be reduced and time can be saved. When calculating the physical model with the help of Matlab mathematical software, some required values and pictures can be obtained. The effects of the two iterative methods can be compared through these data and graphics, Thus, it is verified that Jacobi iterative method has faster convergence speed and better effect. However, for some special equations, only Jacobi iterative method can make them converge, while G-S iterative method does not converge. Of course, there are also cases where Jacobi iterative method does not converge and G-S iterative method converges. Therefore, different iterative methods should be adopted to solve them according to the specific conditions of the equations. It should be noted that the Jacobian iterative method used in this paper is only applicable to matrices with non-zero elements on the diagonal, so it has some limitations. However, these two iterative methods are usually used in solving large sparse linear equations, especially differential equations. In addition, these two iterative methods are often used in engineering calculation, so the study of these two methods is helpful to solve practical problems. Nonlinear spatiotemporal fractional reaction-diffusion equations is discussed, and the solution of the equation in series form is given. This method deals with the nonlinear reaction term well, and obtains a higher accurate approximate solution compared with the nonlinear fractional variable coefficient differential equation. The effectiveness of the method is verified numerically.

Due to the limitation of time and conditions, there are still some deficiencies in this paper: the comparison between time domain solution and frequency domain solution of fractional differential system can be more in-depth; Thirdly, the time domain method proposed in this paper needs to be compared with other time domain methods.

References

[1] Wang Han, Li Huilai. Existence of solutions for a class of nonlinear fractional differential equations [J]. Journal of Jilin University (SCIENCE EDITION), 2016, 54 (005): 1039-1042

[2] Yin Weishi, Han Tao, Yin, et al. Solving nonlinear fractional partial differential equations based on homotopy perturbation sumudu transformation method and sumudu decomposition method [J]. Journal of Fudan University (NATURAL SCIENCE EDITION), 2017, 05 (v.56): 9-14

[3] Wang Yadong, Zhang Xindong. Exp (-Φ(ξ)) Solution [J]. Progress in Applied Mathematics, 2017, 6 (4): 8

[4] Xu Tianliang, fan Xiaomin, Zhang Yuejin. Homotopy analysis solution of nonlinear fractional differential equations [J]. Journal of natural science of Xiangtan University, 2016, 38 (4): 4

[5] Zhou Qin, Yang Yin. Spectral collocation method for solving nonlinear time fractional Klein Gordon equation [J]. Journal of Jilin University (SCIENCE EDITION), 2018, v.56; No. 230 (02): 102-108

[6] Yang Xiaoli, Xu Lei. Numerical solution of fractional differential equations by Chebyshev operator matrix [J]. 2021 (2019-8): 150-153

[7] Yang Xiaoli, Xu Lei. Numerical solution of fractional differential equation by Chebyshev operator matrix [J]. Journal of Suihua University, 2019, v.39 (08): 155-158

[8] Luo Shuanghua, Wang song, Zhang Chengyi, et al. Local polynomial smoothing factor method for numerical solution of Bagley torvik fractional differential equations [J]. Journal of engineering mathematics, 2018

[9] Chen Yanli, song Weixin, Li Hong, et al. Existence of solutions for boundary value problems of a class of Nonlinear Fractional Differential Equations in Banach spaces [J]. Journal of Shanxi Normal University (NATURAL SCIENCE EDITION), 2018, V.32; No.120 (03): 10-16

[10] Mo Jiaqi, Feng Yihu, Wang Weigang. Laminated solutions of Cauchy problem for a class of Nonlinear Nonlocal Singularly Perturbed fractional differential equations [J]. Journal of Sun Yat sen University: Natural Science Edition, 2018

[11] Li Yaohong, Zhang Haiyan. Positive solutions of infinite integral boundary value problems for a class of nonlinear fractional differential equations [J]. Journal of Suzhou University, 2018, 33 (10): 99-101

[12] Liang Shengquan, Yang He. Positive solutions of Neumann boundary value problems for second order ordinary differential equations with variable coefficients [J]. Journal, 2016, 45 (02)

[13] Luo Fang, Wang Zhenfang. Numerical solution of a class of fractional differential equations [J]. Journal of Shanxi Datong University: Natural Science Edition, 2020 (4): 24-26

[14] Wang Zheng, Hu Changliu. Numerical solution of distribution order differential equation using Laplace transform [J]. Journal of natural science of Xiangtan University, 2018, v.40; No. 144 (01): 15-18

[15] Zhu Shuai, Xie Jiaquan, Wu Shiyue. Numerical solution of fractional partial differential equations by Legendre function method [J]. Journal of engineering mathematics, 2018, 035 (005): 570-578

[16] Zhang Wenbiao, Yi Ming. Cauchy problem for a class of abstract nonlinear fractional differential equations [J]. Journal of Applied Mathematics, 2017, 040 (006): 874-882